The Formal Definition of Continuity

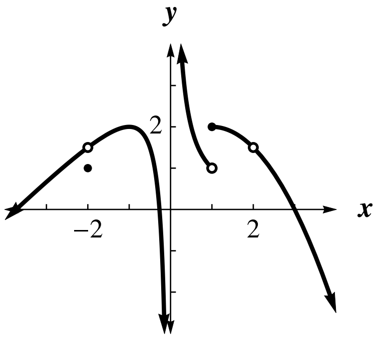
A while back we developed an informal definition of a what a continuous function is: it is one that can be drawn without having to lift your pencil off the paper. While this intuitively makes sense, it is not a very complete definition and it lacks mathematical rigor.

Continuous at a Point

In order to justify that a function, f , is continuous at a point, say $x = a$, three conditions must be met

1. $\lim_{x \to a} f(x)$ exists (which means $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$)
2. $f(a)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

To understand what these conditions really mean, let's look at the graph of an example function and ask the questions: Is it continuous at $x = -2$? Is it continuous at $x = 0$? $x = 1$? $x = 2$?



The answers, of course, are all "no, not continuous" because at each of those points we know the function fails the "pencil-lifting test". But let's see which conditions of continuity are not met at each of those points:

a	$\lim_{x \to a} f(x)$	$f(a)$	$\lim_{x \to a} f(x) = f(a)$?	**Continuous ?**
−2	1.5	1	No	No
0	DNE	DNE	No	No
1	DNE	2	No	No
2	1.5	DNE	No	No

Continuous on an Interval

A function is said to be *continuous over an interval* if it is continuous at each point in the interval. This definition is why functions such as tan(x) are considered to be *continuous over its domain*, even though the actual graph fails the pencil-lifting test. By defining the interval to be the domain of the function, tan(x) becomes continuous on its domain because the function is continuous at each point in that interval.

tan(x)