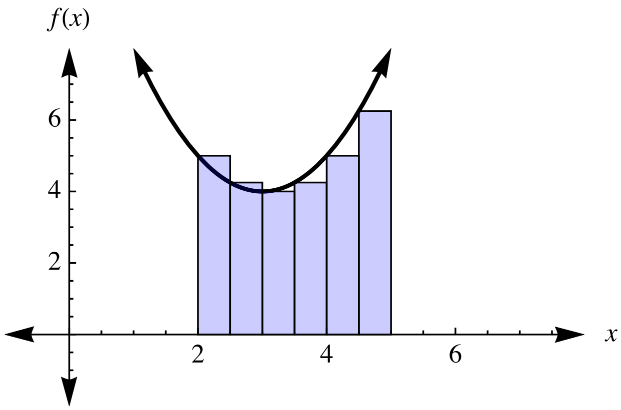
Summation Notation and Area with Left Endpoint Rectangles

Summation notation can be used as a shortcut from writing and evaluating the summation of areas when approximating the area under a curve. Let's consider the function $f(x) = x^2 - 6x + 13$ and approximate the area under the curve for $2 \le x \le 5$ using six left endpoint rectangles, as shown below:



To help us stay organized, let's use a table to summarize the work:

x	$f(x)$	Δx	$A = \Delta x \cdot f(x)$
2.	5.	0.5	2.5
2.5	4.25	0.5	2.125
3.	4.	0.5	2.
3.5	4.25	0.5	2.125
4.	5.	0.5	2.5
4.5	6.25	0.5	3.125

Summing up all the the areas gives us A =14.375 square units.

Note how we could have written this out as an expanded sum:

$$A \approx \frac{1}{2} f(2) + \frac{1}{2}(2.5) + \frac{1}{2} f(3) + \frac{1}{2}(3.5) + \frac{1}{2} f(4) + \frac{1}{2}(4.5)$$

This can be re-written so that an index pattern (going up by 1) is clear:

$$A \approx \frac{1}{2} f\left(2 + \frac{1}{2} \cdot 0\right) + \frac{1}{2}\left(2 + \frac{1}{2} \cdot 1\right) + \frac{1}{2} f\left(2 + \frac{1}{2} \cdot 2\right) + \frac{1}{2}\left(2 + \frac{1}{2} \cdot 3\right) + \frac{1}{2} f\left(2 + \frac{1}{2} \cdot 4\right) + \frac{1}{2}\left(2 + \frac{1}{2} \cdot 5\right)$$

So, if we use an index, x, starting at 0 and going up to 5, we can write the area using summation notation:

$$A \approx \sum_{i=0}^{5} \frac{1}{2} f\left(2 + \frac{1}{2} i\right)$$

Let's take a closer look at the pieces to this summation. First, here are how the width and height of each rectangle is represented.

$$A \approx \sum_{i=0}^{5} \underbrace{\frac{1}{2}}_{\text{Width}} \cdot \underbrace{f\left(2 + \frac{1}{2} i\right)}_{\text{Height}}$$

Here is how the starting x value of 2 and the width increment (also known as the "step size") of 1/2 are represented:

$$A \approx \sum_{i=0}^{5} \frac{1}{2} \cdot f\left(2 + \frac{1}{2}\, i\right)$$

Starting x (2) Width ($\frac{1}{2}$)

Example

Using the function $f(x) = x^2 - 6x + 13$, write an expression using sigma notation to approximate the area under the curve for $4 \le x \le 12$ using 24 left endpoint rectangles of equal width.

Solution

From the information given in the question, we know that

1. The starting x value is 4 and
2. The width of each rectangle is $(12 - 4) / 24 = 8 / 24 = 1 / 3$

Using the pattern shown above, we can write the expression for left endpoint rectangles:

$$A \approx \sum_{i=0}^{23} \frac{1}{3} f\left(4 + \frac{1}{3}\, i\right)$$

If you are curious, here is what the graph looks like: