Rational Functions: Holes versus Asymptotes

Asymptotes

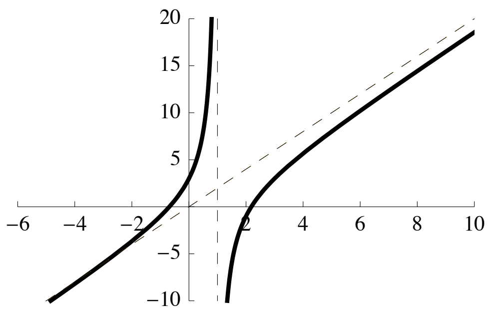
Because the denominator of a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

cannot be zero, $q(x) \neq 0$, x values that make the denominator zero cannot be in the domain of the function. For example, let's consider the rational function:

$$f(x) = \frac{1}{x-1}\left(2\,x^2 - 3\,x - 3\right)$$

If we graph this function we get:

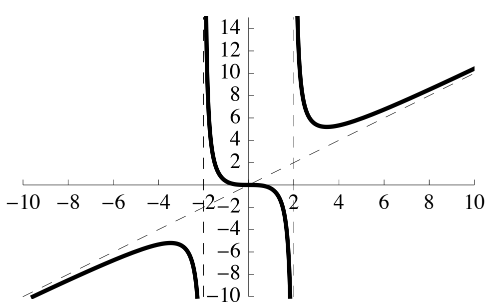


Notice how when $x = 1$, the denominator is 0 and so 1 cannot be in the domain of this function. This is what causes the vertical asymptote at $x = 1$:

$$f(1) = \frac{1}{(1)-1}\left(2\,(1)^2 - 3\,(1) - 3\right) = \frac{2-3-3}{0} = \frac{-4}{0} = \text{Undefined}$$

Here is another example,

$$f(x) = x^3 \big/ \left(x^2 - 4\right)$$



This function has two vertical asymptotes: one at $x = -2$ and another at $x = 2$, because:

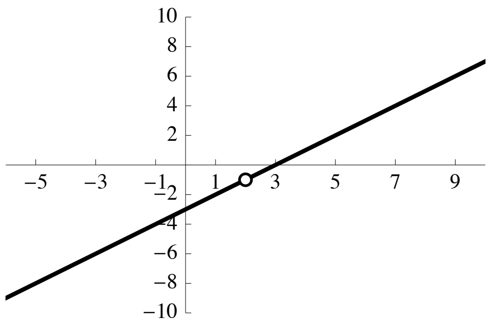
$$f(-2) = \frac{(-2)^3}{(-2)^2 - 4} = \frac{-8}{4-4} = \frac{-8}{0} = \text{Undefined} \qquad \text{and}$$

$$f(2) = \frac{(2)^3}{(2)^2 - 4} = \frac{8}{4-4} = \frac{8}{0} = \text{Undefined}$$

Holes

Now let's consider another rational function:

$$f(x) = \frac{x^2 - 5x + 6}{x - 2}$$



In this case there is not a vertical asymptote when the denominator is zero ($x = 2$) but there is a "hole" in the function at that point (which is also called a "point of discontinuity"). To understand why, let's evaluate $f(2)$:

$$f(2) = \frac{1}{(2) - 2}\left((2)^2 - 5\,(2) + 6\right) = \frac{4 - 10 + 6}{0} = \frac{0}{0}$$

In this case, when the denominator is zero, the numerator is also zero. This is because $x - 2$ is a factor of $x^2 - 5x + 6$:

$$f(x) = \frac{1}{x - 2}\left(x^2 - 5x + 6\right) = \frac{(x - 2)\,(x - 3)}{(x - 2)}$$

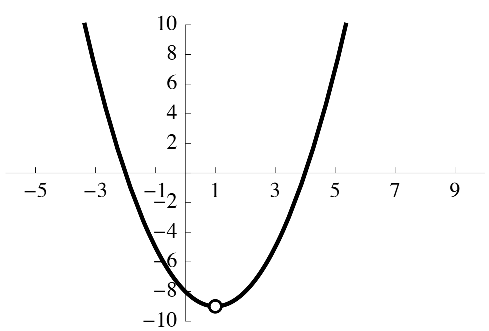
For all values of x, except at $x = 2$ the function behaves as if it were $f(x) = x - 3$ (and hence the graph of a line) but at $x = 2$, there is no point on the line because $0/0$ is said to be *indeterminate*; that is, it's value cannot be determined:

$$\frac{0}{0} = \text{Indeterminate}$$

Indeterminate points of a function create holes (or points of discontinuity) when the function is graphed.

Here's another example of a function with a hole (written in factored form):

$$f(x) = \frac{(x - 4)\,(x - 1)\,(x + 2)}{x - 1}$$



When $x \neq 1$, this function behaves like the quadratic function:

$$f(x) = (x - 4)\,(x + 2) = x^2 - 2x - 8$$

But when $x = 1$, we get:

$$f(1) = \frac{(1 - 3)\,(1 - 1)\,(1 + 2)}{(1 - 1)} = \frac{-2 \cdot 0 \cdot 3}{0} = \frac{0}{0} = \text{Indeterminate}$$

so there is a hole in the function at $x = 1$.