Composite Functions

The Cost of Carpet

Imagine there is a square room with lengths of 10 feet by 10 feet, and that we want carpet the room. Let’s also assume that the cost of the carpet is simply $5 per square foot. If we want to calculate how much it will cost to carpet the room there are two steps we need to make:

1. Find the area of the room (in square feet):

   Area = $10 \times 10 = 100$ square feet

2. Find the cost of that amount of area:

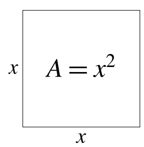
   Cost = $5 \times 100 = \$500$

Thinking about Functions

While this may seem to be a very simple problem, let’s re-think it in terms of functions.

Step 1

In our first step, we actually used an area function to find the area of a square given it's side length, which we can call x:



If we use proper function notation, we can say that: $A(x) = x^2$.

Step 2

In our second step, we used a cost function to find how much it cost to by carpet for a given area:

$$C = 5 \cdot a$$

If we used proper function notation, we can say that $C(a) = 5\,a$.

Putting the Steps Together

Since $a = x^2$ in this simple example, we can write:

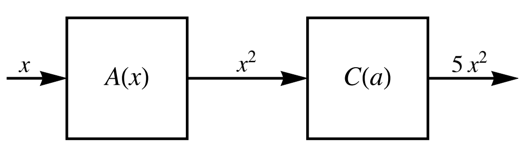
$$C = 5\,x^2$$

or, more formally,

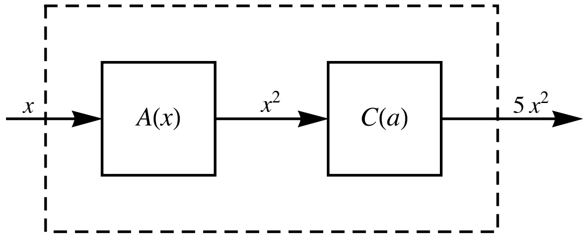
$$C(x) = 5\,x^2$$

Thinking Visually with Function Machines

We can take what we did with functions and describe it using function machines:



Notice how the output of the first machine becomes the input of the second machine. We could actually think of this as one “big” machine that has x as an input and $5\,x^2$ as an output:



The function that we get when we combine functions in this way is known as a *composite function*. Let's call the composite function we created $d(x)$ where "d" means dollars:

$$d(x) = 5\,x^2$$

Composite Function Notation

While it's easy to show how $d(x)$ was created by combining $A(x)$ and $C(a)$ with a diagram, mathematicians have developed a short-hand notation for describing a composite function:

$$d(x) = C(A(x))$$

This is read as "d of x is equal to C of A of x".

What this means is actually quite simple if we refer back to our diagram: For a square room with side lengths x, we use the output from our area function, $A(x)$, and use it as input to our cost function, $C(x)$, and that gives us how many dollars we have to spend.

Example

If $f(x) = 2\,x + 3$ and $g(x) = -3\,x^2 + 5$, evaluate $f(g(1))$and $g\,(f(1))$.

To solve $f(g(1))$, we always start with the inner function

$$g(4) = -3\left(1^2\right) + 5 = -3\,(1) + 5 = 2$$

And then we plug that into $f(x)$:

$$f(2) = 2\,(2) + 3 = 4 + 3 = 7.$$

To solve $g(f(1))$, we again start with the inner function

$$f(1) = 2\,(1) + 3 = 5$$

And then plug that into $g(x)$:

$$g(5) = -3\left(5^2\right) + 5 = -75 + 5 = -70$$

Example (Challenge)

Rather than evaluating f and g (from the previous example) separately, we can first find the composite function and then plug in the the value of 1:

$$f(g(x)) = f\left(-3\,x^2 + 5\right) = 2\left(-3\,x^2 + 5\right) + 3 = -6\,x + 10 + 3 = -6\,x + 13$$

So

$$f(g(1)) = -6\,(1) + 13 = 7$$

And

$$g(f(x)) = g(2\,x + 3) = -3\,(2\,x + 3)^2 + 5 = -3\left(4\,x^2 + 12\,x + 9\right) + 5 = -12\,x^2 - 36\,x - 22$$

So

$$g(f(1)) = -12\left(1^2\right) - 36\,(1) - 22 = -70$$