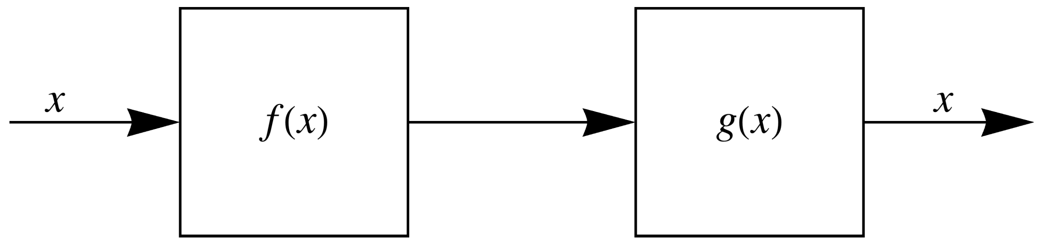
Inverse Functions

The Definition

Inverse functions are functions that “undo” each other or “cancel” each other out.



In other words, if we put some value x into $f(x)$ and then take the output of $f(x)$ and put it into $g(x)$, the output from $g(x)$ will be x.

Simply put: $g(x)$ undoes what $f(x)$ does.

Composite Function Notation

Using composite function notation, we can write the formal mathematical definition of inverse functions:

$$g(f(x)) = f(g(x)) = x$$

Inverse functions are very common and everyone has seen them before. For example:

$f(x) = 2x$ and $g(x) = x/2$

$f(x) = x^2$ and $g(x) = \sqrt{x}$ (for $x \geq 0$)

$\sin(x)$ and $\arcsin(x)$

Example Problem

Show that $f(x) = 4x + 3$ and $g(x) = \frac{x-3}{4}$ are inverse functions.

Solution

To show that f and g are inverse functions we need to show that $g(f(x)) = x$ and $f(g(x)) = x$:

$$g(f(x)) = g(4x + 3) = \frac{1}{4}((4x + 3) - 3) = \frac{1}{4}(4x + 3 - 3) = \frac{4x}{4} = x$$

and

$$f(g(x)) = f\left(\frac{x-3}{4}\right) = 4\left(\frac{x-3}{4}\right) + 3 = \frac{4}{4}(x - 3) + 3 = x - 3 + 3 = x$$

So f and g are inverse functions!

Inverse Function Notation

The inverse of a function, such as $f(x)$, is denoted by $f^{-1}(x)$. This type of notation you have probably seen for the arcsine, arccosine and arctangent functions on your calculator:

$\arcsin(x) = \sin^{-1}(x)$ $\arccos(x) = \cos^{-1}(x)$ $\arctan(x) = \tan^{-1}(x)$

Using this notation, we can write:

$f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x$

Finding Inverse Functions

Let's say we have a function $f(x) = 2x - 5$ and we want to find its inverse. How can we take the expression $2x - 5$ and reduce it to x?

1. Add 5 to $2x - 5$ to get $2x$.

2. Divide $2x$ by 2 to get x.

Our inverse function is $g(x) = \frac{x+5}{2}$!

Notice how this is the same as "isolating x":

$$f(x) = 2x - 5$$
$$f(x) + 5 = 2x$$
$$\frac{f(x) + 5}{2} = x$$

Example Problem: Finding an Inverse Function

Find the inverse of $f(x) = 2x^2 - 3$

Solution

Rearrange to get x by itself:

$$f(x) = 2x^2 - 3$$
$$f(x) + 3 = 2x^2$$
$$\frac{f(x) + 3}{2} = x^2$$
$$\sqrt{\frac{f(x) + 3}{2}} = x$$

So our inverse function is $g(x) = \sqrt{\frac{x+3}{2}}$

Inverse Functions and "Swapping *x* and *y*"

We can get an inverse function by "swapping x and y". For example, find the inverse of $y = x^2 + 2$. Isolate x:

$$y = x^2 + 2$$
$$y - 2 = x^2$$
$$\sqrt{y - 2} = x$$

And then swap x and y:

$$y = \sqrt{x - 2}$$

An inverse function causes all the points on the original function to get swapped: the x values become the y values and the y values become the x values.

Inverse Functions and Reflecting across $y = x$

Recall your rigid motion work from Grade 9: swapping x and y is exactly what happens when we reflect across the line $y = x$.

We can visualize an inverse function as being the reflection of the original function across the line $y = x$. Here's the plot of $x^2 + 2$ and $\sqrt{x - 2}$: