The Formal Definition of Continuity

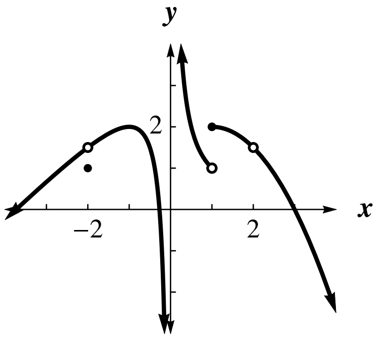
A while back we developed an informal definition of a what a continuous function is: it is one that can be drawn without having to lift your pencil off the paper. While this intuitively makes sense, it is not a very complete definition and it lacks mathematical rigor.

Continuous at a Point

In order to justify that a function, f , is continuous at a point, say $x = a$, three conditions must be met

1. $\lim_{x \to a} f(x)$ exists (which means $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$)
2. $f(a)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

To understand what these conditions really mean, let's look at the graph of an example function and ask the questions: Is it continuous at $x = -2$? Is it continuous at $x = 0$? $x = 1$? $x = 2$?



The answers, of course, are all "no, not continuous" because at each of those points we know the function fails the "pencil-lifting test". But let's see which conditions of continuity are not met at each of those points:

a	$\lim_{x \to a} f(x)$	$f(a)$	$\lim_{x \to a} f(x) = f(a)$?	Continuous ?
−2	1.5	1	No	No
0	DNE	DNE	No	No
1	DNE	2	No	No
2	1.5	DNE	No	No

Continuous on an Interval

A function is said to be *continuous over an interval* if it is continuous at each point in the interval. This definition is why functions such as $\tan(x)$ are considered to be *continuous over its domain*, even though the actual graph fails the pencil-lifting test. By defining the interval to be the domain of the function, $\tan(x)$ becomes continuous on its domain because the function is continuous at each point in that interval.