Solving Logarithmic Equations

A logarithmic equation, as the name implies, is one that has at least one term with a log. A very simple logarithmic equation would be:

$$\log(x) = 11$$

As you might have guessed, since we use logs to solve exponential equations, we use exponents to solve logarithmic equations.

Example 1

Let's solve the previous equation:

$$\log(x) = 11$$
$$10^{\log(x)} = 10^{11}$$
$$x = 10^{11}$$

Example 2

In some logarithmic equations, we need to solve for the base:

$$\log_b(243) = 5$$

There are multiple ways of solving this. One way is to rewrite the equation as:

$$b^5 = 243$$

We can now use exponent properties to solve for b:

$$\left(b^5\right)^{1/5} = 243^{1/5}$$
$$b = 243^{1/5}$$

We could use a calculator to determine that $b = 3$.

Another way of solving this is to use the Change of Base Rule:

$$\log_b(243) = 5$$
$$\frac{\log(243)}{\log(b)} = 5$$
$$\log(b) = \frac{\log(243)}{5}$$
$$b = 10^{\log(243)/5} = 3$$

Example 3

When there are multiple terms, it is often useful to condense the logarithms first (which is why we learned how to condense logarithms). Here is an example:

$$\ln(8\,w) + \ln(2\,w) = 12$$
$$\ln\left(16\,w^2\right) = 12$$
$$e^{\ln\left(16\,w^2\right)} = e^{12}$$
$$16\,w^2 = e^{12}$$
$$w^2 = \frac{e^{12}}{16}$$
$$w = \pm\sqrt{\frac{e^{12}}{16}} = \pm\frac{e^6}{4} = \pm 100.8572$$

Example 4

Sometimes we will encounter expressions as the argument of the log function. Here is an example:

$$\log_4(8x-4) = 2$$

We solve these by "undoing" the log with an exponentiation and then simplifying:

$$4^{\log_4(8x-4)} = 4^2$$
$$8x-4 = 16$$
$$8x = 20$$
$$x = \frac{5}{2}$$

Example 5

Solve for p:

$$\log_2 p - \log_2(3p-2) = \log_2 9$$

Let's begin by condensing the left side :

$$\log_2\left(\frac{p}{3p-2}\right) = \log_2 9$$

Next, get rid of the logs using exponentiation:

$$\frac{p}{3p-2} = 9$$

Finally, get p by itself:

$$p = 9(3p-2)$$
$$p = 27p-18$$
$$26p = 18$$
$$p = \frac{9}{13}$$

[Challenge] Example 6

It is possible to create logarithmic and exponential equations that are difficult or impossible to solve algebraically. Here is an example:

$$\ln(x) + e^x = 0$$

One way to get an approximate answer to this equation is to graph it and estimate where it crosses the x-axis. The accuracy can be improved by "guess and check" using a spreadsheet or a program. After a little effort, an approximate solution can be found:

$$x = 0.269874$$

Another example of challenging equation is:

$$\log_x(5) = x$$

This we can rewrite as

$$x^x = 5$$

and then take the natural logs of both sides to get

$$x \ln x = \ln 5$$