Why Do Logarithms Matter?

Exponential Growth and Decay

In the past, you have worked with exponential growth and decay. For example, you might have come up with a formula that governs the growth of bacteria in petri dish, such as:

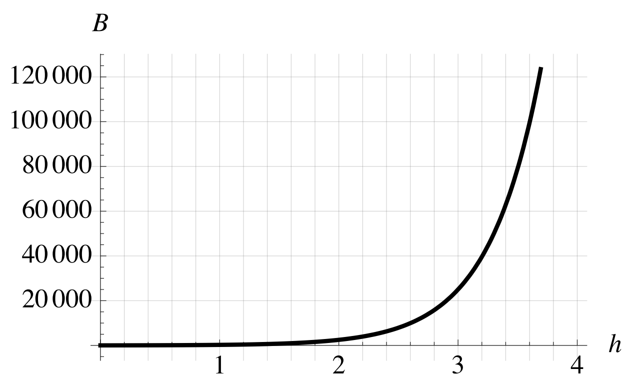
$$B = 25 \cdot 10^h$$

where B is the number of bacteria after h hours. A table helps visualize this growth:

h	0	1	2	3	4	5
B	25	250	2500	25 000	250 000	2 500 000

One of the questions we could ask about this growth is: When does the number of bacteria equal 100,000?

From the table, we can see that it happens some time between 3 and 4 hours, but it's difficult to be precise. If we were to graph the function, we could make a better guess:



From the graph, it seems that there are 100,000 bacteria after 3.6 hours. But how can we get an exact answer (or an answer that is precise to as many decimal places as we want)? The answer is logarithms.

We want to solve the following equation for h:

$$100\,000 = 25 \cdot 10^h$$

As a first step let's get the exponent by itself by first dividing both sides of the equation by 25:

$$10^h = 4000$$

Since the base of our exponent is 10, let's now take the logarithm of both sides using base 10:

$$\log_{10}\left(10^h\right) = \log_{10}(4000)$$

Since "logs undo exponents", our equation becomes:

$$h = \log_{10}(4000)$$

If we use a calculator to find the log (base 10) of 4000 we get:

$$h = 3.602059991$$

Our guess from the graph was pretty close but the logarithm function gives us a much more precise answer to the question.

This is just one example of where logs are used for problem-solving. In general, whenever we are dealing with exponential functions, we often use logarithmic functions to solve problems.

Sound Volume

Without delving too deeply into the physics of sound, sound intensity is measured as power per unit area which has units of:

$$\text{watts} \Big/ \text{meter}^2$$

The human threshold of hearing (ToH) is $10^{-12}\ W / m^2$. The threshold of hearing pain is $10\ W / m^2$.

Since the range of intensities that the human ear can detect is so large, the scale that is frequently used by physicists to measure intensity is a scale based on powers of 10, which is a logarithmic scale. The scale for measuring intensity is known as a “decibel scale”. On this scale, the threshold of hearing (ToH) is assigned an intensity of 0 Bels. A sound that has 10 times the intensity ($10 \cdot 10^{-12} = 10^{-11}\ W / m^2$) is assigned an intensity of 1 Bel. A sound that has intensity of $100 = 10 \times 10 = 10^2$ times the ToH is assigned an intensity of 2 Bels, and so on. A sound with an intensity of 5 Bels would have an intensity of $10^5 = 100\,000$ times greater than the threshold of hearing.

In practice, the units of Bels are not used; instead, measurements are done in $1 \Big/ 10^{\text{th}}$of a Bel, which is called a decibel (abbreviated dB). 1 bel = 10 dB, 2 Bels = 20 dB, and so forth. The following table summarizes some sound intensities.

Source	Intensity W/m^2	Intensity Level (dB)	No. of Times Greater than ToH
Threshold of Hearing (TOH)	$1. \times 10^{-12}$	0	10^0
Rustling Leaves	$1. \times 10^{-11}$	10	10^1
Whisper	$1. \times 10^{-10}$	20	10^2
Normal Conversation	$1. \times 10^{-6}$	60	10^6
Busy Street Traffic	$1. \times 10^{-5}$	70	10^7
Vacuum Cleaner	$1. \times 10^{-4}$	80	10^8
Large Orchestra	6.3×10^{-3}	98	$10^{9.8}$
iPod at Maximum Level	$1. \times 10^{-2}$	100	10^{10}
Front Rows of Rock Concert	$1. \times 10^{-1}$	110	10^{11}
Threshold of Pain	$1. \times 10^1$	130	10^{13}
Military Jet Takeoff	$1. \times 10^2$	140	10^{14}
Instant Perforation of Eardrum	$1. \times 10^4$	160	10^{16}

A mosquito’s buzz is often rated with a decibel rating of 40 dB. Normal conversation is often rated at 60 dB. How many times more intense is normal conversation compared to a mosquito’s buzz?

A mosquito’s buzz is 10^4times greater than the ToH and the normal conversation is 10^6times greater, so the ratio is:

$$\frac{10^6}{10^4} = 10^2 = 100$$

Therefore, a normal conversation is 100 times more intense than a mosquito’s buzz. Another way of getting to this answer is to note that the difference in intensity is 20 dB or 2 Bels:

$$10^2 = 100$$

Acids, Bases and the pH scale

With going to deep into chemistry, solutions can be characterized as being acidic, neutral or basic. To help understand this, let’s focus on water. A water molecule as one oxygen atom and two hydrogen atoms: H_2O. It is possible for a water molecule to split into an hydroxide ion, OH^-, and a hydrogen ion, H^+(which will then join a water molecule to become a hydronium ion, H_3O^+, but for simplicity are still referred to as hydrogen ions). Water that has an equal number of hydrogen and hydroxide ions is neutral: it is neither acid nor basic.

An acid is a solution that donates hydrogen ions: if an acid is combined with neutral water, the resulting water will have more hydrogen ions that hydroxide ions.

A base is a solution that donates hydroxide ions: if a base is combined with neutral water, the resulting water will have more hydroxide ions that hydrogen ions.

Here is an interesting fact: a strongly acidic solution can have one hundred trillion (that's 100,000,000,000,000) times more hydrogen ions than a strongly basic (alkaline) solution. And likewise, a strongly basic (alkaline) solution can have a hundred trillion more hydroxide ions than a strongly acidic solution. Given the size of these numbers, they are not practical to use to describe the acidity or alkalinity of a solution; instead, a logarithmic scale (base 10) is used. This scale is known as the pH scale: each one-unit change in pH corresponds to a 10-fold change in the hydrogen ion concentration. The log (base 10) of one hundred trillion is

$$\log_{10}(100\,000\,000\,000\,000) = \log_{10}\left(10^{14}\right) = 14$$

While the pH scale is essentially open-ended, 14 is considered the upper bound on how basic a solution can be. If 0 is considered the most acidic solution, then a neutral solution (halfway between the most acidic and the most basic) would have a pH of 7.

To be really precise about all of this, the formal definition of a pH value is

$$\text{pH} = -\log_{10}[\text{H}^+]$$

which to a chemist means that the pH is the negative value of the common log of the molar concentration of hydrogen ions.

The following table summarizes some pH values:

pH Value	H^+ Concentration Relative to Pure Water	Example
0	10 000 000	battery acid
1	1 000 000	gastric acid
2	100 000	lemon juice,vinegar
3	10 000	orange juice,soda
4	1000	tomato juice,acid rain
5	100	black coffee,bananas
6	10	milk,saliva
7	1	pure water
8	0.1	sea water,eggs
9	0.01	baking soda
10	0.001	Great Salt Lake, milk of magnesia
11	0.000 1	ammonia solution
12	0.000 01	soapy water
13	0.000 001	bleach,oven cleaner
14	0.0000 001	liquid drain cleaner

Earthquakes

The Richter Scale, used to measure the energy of an earthquake, is a logarithmic scale (base 10). That means that an earthquake that measures 5 on the Richter scale has 10 times more energy than an earthquake that measures a 4. The largest earthquake ever recorded was a 9.5 quake in Chile in 1960. One of the more recent California earthquakes had a magnitude of 7.1. This raises the question: how much more energy is there in a 9.5 quake than a 7.1?

We know that a quake of 9 is 100 times stronger than a quake of 7 because:

$$\frac{10^9}{10^7} = 10^{9-7} = 10^2 = 100$$

So, for earthquakes of magnitude 9.5 and 7.1 the ratio is:

$$\frac{10^{9.5}}{10^{7.1}} = 10^{9.5-7.1} = 10^{2.4} = 251.189$$

So the Chilean earthquake was over 250 times stronger than the recent California earthquake.

Here's another type of question: If the Chilean earthquake had a magnitude of 9.5, what magnitude of earthquake is 500 times less? In terms of an equation, we are asking to find m:

$$\frac{10^{9.5}}{10^{m}} = 500$$

If we rearrange this equation, we get

$$10^{m} = \frac{10^{9.5}}{500}$$

If we take the logarithm of both sides, we can get our answer (using a calculator, of course):

$$\log_{10}(10^{m}) = \log_{10}\left(10^{9.5} / 500\right)$$
$$m = 6.801$$

Large-Magnitude Math

In some types of math, one ends up working with extremely large intermediate values that would cause an overflow on most calculators (this is typical in some types of probability and combinatorics problems). For example:

$$500!$$

Recall that the exclamation sign means multiply:

$$500! = 500 \cdot 499 \cdot 498 \cdot 497 \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

As we will prove soon, logarithms have some very interesting properties; in fact, we can write:

$$\log_{10}(500!) = \log_{10} 500 + \log_{10} 499 + \log_{10} 498 + \log_{10} 497 + \cdots + \log_{10} 3 + \log_{10} 2 + \log_{10} 1$$

We still have 500 additions to do, but if we write a simple computer program (or use a spreadsheet), we can quickly determine that

$$\log_{10}(500!) = 1134.086409$$

If we undo the logarithm with an exponent, we get our answer

$$500! = 10^{1134.086409} = 10^{1134+0.086409} = 10^{1134} \cdot 10^{0.086409} = 1.220136826 \times 10^{1134}$$

That's a very big number.