Expanding Logarithmic Expressions

Using the properties of logarithms, we can manipulate logarithmic expressions. One form of manipulation is to expand an expression. In these notes, we explore various types and combinations of expansions.

Using the Power Property

If we have a logarithmic expression such as

$$\log\left(5^3\right)$$

we can use the Power Property to expand the expression to

$$\log\left(5^3\right) = 3\log(5)$$

We can also, of course, do this with variables as well as numbers:

$$\log(5^p) = p\log(5)$$
$$\log\left(g^{20}\right) = 20\log(g)$$
$$\log\left(x^{-2}\right) = -2\log(x)$$
$$\log(p^q) = q\log(p)$$

Using the Product Property

If we have a logarithmic expression such as

$$\log(100\,x)$$

we can use the Product Property to expand the expression to

$$\log(100\,x) = \log(100) + \log(x) = \log\left(10^2\right) + \log(x) = 2 + \log(x)$$

Using the Quotient Property

If we have a logarithmic expression such as

$$\log_3\left(\frac{81}{w}\right)$$

we can use the Quotient Property to expand the expression to

$$\log_3\left(\frac{81}{w}\right) = \log_3(81) - \log_3(w) = 4 - \log_3(w)$$

Converting Radical Signs to Exponentials

If the expression has a radical sign such as

$$\log_2\left(\sqrt[4]{x}\right)$$

the first step is to convert the radical to an exponent:

$$\log_2\left(\sqrt[4]{x}\right) = \log_2\left(x^{1/4}\right)$$

and then use the Power Property:

$$\log_2\left(\sqrt[4]{x}\right) = \log_2\left(x^{1/4}\right) = \frac{1}{4}\log_2(x)$$

Using Combinations of Properties

Some expansions require the use of more than one property. For example,

$$\log_b\left(\frac{x^2\,y^3}{z^4}\right)$$

There is often more than one way to expand, though one has to be careful about when to use the Power Property. In this example, it easiest to begin with the Quotient Property:

$$\log_b\left(\frac{x^2\,y^3}{z^4}\right) = \log_b\left(x^2\,y^3\right) - \log_b\left(z^4\right)$$

We can now use the Product Property and the Power Property to continue the expansion:

$$\log_b\left(\frac{x^2\,y^3}{z^4}\right) = \log_b\left(x^2\,y^3\right) - \log_b\left(z^4\right) = \log_b\left(x^2\right) + \log_b\left(y^3\right) - 4\log_b(z)$$

Finally, we use the Power Property again:

$$\log_b\left(\frac{x^2\,y^3}{z^4}\right) = 2\log_b(x) + 3\log_b(y) - 4\log_b(z)$$

Sometimes it's easier to simplify an expression before using the properties. For example,

$$\log\left(\frac{x^7}{\sqrt{x^5(x^2 - 2\,x + 1)}}\right)$$

In one step, by using exponent properties, distribution and the Quotient Property, this can be written as

$$\log\left(x^7\right) - \log\left(x^7 - 2\,x^6 + x^5\right)^{1/2}$$

We can finish the expansion using the Quotient Rule:

$$7\log(x) - \frac{1}{2}\log\left(x^7 - 2\,x^6 + x^5\right)$$

Note that we cannot expand this any further (we cannot distribute the log function over addition or subtraction).