Solving Exponential Equations

As mentioned in an earlier set of notes, logarithms are useful for solving exponential equations. Now that we have an understanding of both the properties of exponents and the properties of logarithms, we can solve more complex exponential equations.

Example 1

Let's start with a simpler equation:

$$4^{2x-3} = 8$$

This first step is to take the logarithm of both sides. While in this example you might be tempted to use base 4 or base 8 for the logarithms, scientists and engineers tend to stick with common or natural logs. Let's use common logs:

$$\log\left(4^{2x-3}\right) = \log(8)$$

We now use properties of logs to expand and then simplify:

$$(2x-3)\log(4) = \log(8)$$

$$2x-3 = \frac{\log(8)}{\log(4)}$$

$$2x = \frac{\log(8)}{\log(4)} + 3$$

$$x = \frac{\log(8)}{2\log(4)} + \frac{3}{2}$$

At this point, we have an exact expression for x, so we could stop simplifying and use a calculator to get our answer or we could use our knowledge of logs to simplify further. Since $2^2 = 4$ and $2^3 = 8$, we can write:

$$x = \frac{\log(8)}{2\log(4)} + \frac{3}{2} = \frac{\log(2^3)}{2\log(2^2)} + \frac{3}{2} = \frac{3\log(2)}{4\log(2)} + \frac{3}{2} = \frac{3}{4} + \frac{3}{2} = \frac{3}{4} + \frac{6}{4} = \frac{9}{4}$$

In this example, we were able to get a fraction for an answer:

$$x = 9/4$$

Example 2

Oftentimes we will not be so lucky and get such nice answers. Here's another example:

$$3^{4x+2} = 5^{-2x+1}$$

Our first step is to take the log of both sides (let's use natural log this time):

$$\ln\left(3^{4x+2}\right) = \ln\left(5^{-2x+1}\right)$$

Once again, we use the properties of logs to expand and simplify:

$$(4x+2)\ln(3) = (-2x+1)\ln(5)$$

$$4x\ln(3) + 2\ln(3) = -2x\ln(5) + \ln(5)$$

$$4x\ln(3) + 2x\ln(5) = \ln(5) - 2\ln(3)$$

$$2x(2\ln(3) + \ln(5)) = \ln(5) - 2\ln(3)$$

$$x = \frac{\ln(5) - 2\ln(3)}{2\,(2\ln(3) + \ln(5))}$$

Once again we might be tempted to stop and use a calculator at this point, but it's always worth trying to simplify as much as possible, even if it only makes using a calculator easier (and less error-prone)

$$x = \frac{\ln(5) - 2\ln(3)}{2\,(2\ln(3) + \ln(5))} = \frac{\ln(5) - \ln(9)}{2\,(\ln(9) + \ln(5))} = \frac{\ln(5/9)}{\ln(81) + \ln(25)} = \frac{\ln(5/9)}{\ln(81 \cdot 25)}$$

Using a calculator we get:

$$x = -0.077204988$$

Example 3

When the exponential equations uses e as a base, it is always best to solve using natural logarithms. It is also good practice to simplify the expression as much as possible before taking the log of both sides. Here's an example:

$$7\,e^{2x} + 4 = 21$$
$$7\,e^{2x} = 17$$
$$e^{2x} = 17/7$$
$$\ln\left(e^{2x}\right) = \ln(17/7)$$
$$2\,x = \ln(17/7)$$
$$x = \frac{\ln(17/7)}{2} = 0.4436516$$

[Challenge] Example 4

When the exponential function has exponents with terms with powers greater than 1, it is possible for there to be more than one answer. Here's an example:

$$5\,e^{x^2-2} = 30$$
$$e^{x^2-2} = 6$$
$$\ln\left(e^{x^2-2}\right) = \ln(6)$$
$$x^2 - 2 = \ln(6)$$
$$x^2 = 2 + \ln(6)$$
$$x = \pm\sqrt{2 + \ln(6)} = \pm 1.94724407$$