Sketching Rational Functions

To sketch the graph of a rational function, there a few steps that we follow. Let's illustrate with an example:

Example

Graph the rational function:

$$f(x) = \frac{x^2 + 4x + 5}{x + 1}$$

Step 1: Find the *x* values that are not in the domain of the function

Since we cannot divide by zero, the x values that are not in the domain of the function are the ones that make the denominator zero:

$$x + 1 = 0$$
$$x = -1$$

So, at $x = -1$, there is either a hole or a vertical asymptote.

Step 2: Determine where there are holes and where there are vertical asymptotes

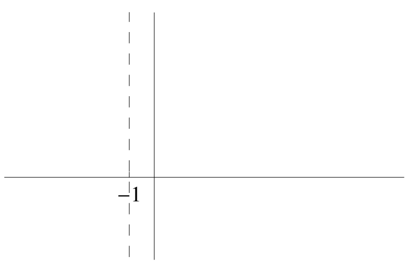
Using the x values obtained in Step 1, we evaluate the function at those values to see if the function is Undefined at that value (and hence a vertical asymptote) or Indeterminate (and hence a hole). For our example:

$$f(-1) = \frac{(-1)^2 + 4(-1) + 5}{(-1) + 1} = \frac{1 - 4 + 5}{0} = \frac{2}{0} = \text{Undefined}$$

Since the function is undefined at $x = -1$ it has a vertical asymptote with equation $x = -1$.

Step 3: Sketch the vertical asymptote(s), if any.

Create a simple sketch of the vertical asymptote. Be sure to label the graph so it is clear where it lies.



Step 4: Determine how the function approaches the asymptote from the right

Create an In-Out table of function values near the vertical asymptote. We are trying to determine whether:

As $x \to -1^+$, $y \to +\infty$ or
As $x \to -1^+$, $y \to -\infty$

x	$f(x) = \frac{x^2 + 4x + 5}{x + 1}$
–0.9	22.1
–0.92	27.08
–0.94	35.3933
–0.96	52.04
–0.98	102.02

From this table we can see that as $x \to -1^+$, $y \to +\infty$.

Step 5: Determine how the function approaches the asymptote from the left

Create an In-Out table of function values near the vertical asymptote. We are trying to determine whether:

As $x \to -1^-, \ y \to +\infty$ or
As $x \to -1^-, \ y \to -\infty$

x	$f(x) = \frac{x^2+4x+5}{x+1}$
−1.1	−18.1
−1.08	−23.08
−1.06	−31.3933
−1.04	−48.04
−1.02	−98.02

From this table we can see that as $x \to -1^-, \ y \to -\infty$. This means that the graph will be moving down towards negative infinity on the left-hand side of the vertical asymptote.

Step 6: Simplify the function to get the end behavior

To see what happens as the x values approach positive infinity and negative infinity, we simplify the function using Wolfram Alpha (or a similar app), factoring (Honors) or polynomial division (Honors).

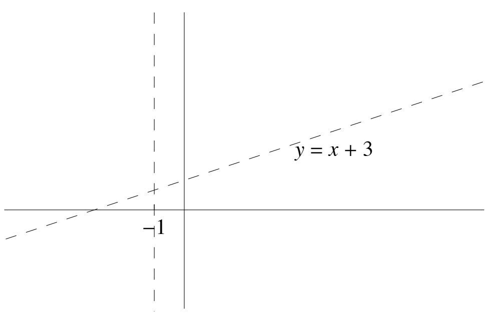
Using Wolfram Alpha, we can see that

$$f(x) = \frac{x^2 + 4x + 5}{x + 1} = x + 3 + \frac{2}{x + 1}$$

This means that as x gets really large (as $x \to +\infty$) or really small (as $x \to -\infty$), the function will have an end behavior of

$$f(x) = x + 3$$

which is a line with a slope of 1 and a y-intercept of 3. We can add this line (also known as a *slant asymptote*) to our graph:



Step 7: Roughly sketch the graph

Since the function will approach the asymptotes, we can roughly show its behavior without having to spend a lot of time graphing points: