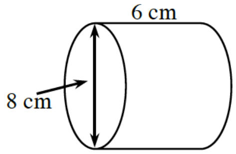
Homework Solutions #3

1. Calculate the volume of each of the following solids.

a.

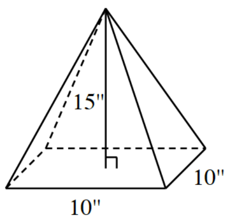


For a cylinder, $V = \pi r^2 h$, where r is the radius and h is the height.

For this cylinder, $r = 8/2 = 4$ cm and $h = 6$ cm:

$$V = \pi r^2 h = \pi 4^2 \cdot 6 = 96\pi \text{ cm}^3$$

b.

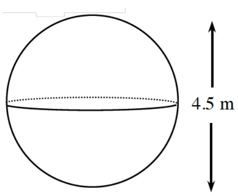


For a pyramid, $V = \frac{1}{3} a h$, where a is the area of the base and h is the height.

For this pyramid, the base is a square with side length $s = 10$ in and $h = 15$ in.

$$V = \frac{1}{3} a h = \frac{1}{3} s^2 h = \frac{1}{3} \cdot 10^2 \cdot 15 = 500 \text{ in}^3$$

c.

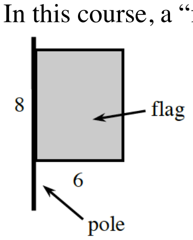


For a sphere, $V = \frac{4}{3} \pi r^3$, where r is the radius.

For this sphere, $r = 4.5/2 = 9/4$ meters.

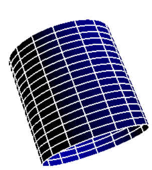
$$V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi \left(\frac{9}{4}\right)^3 = \frac{4}{4^3} \cdot \frac{9}{3} \cdot 9^2 \pi = \frac{3 \cdot 81}{16} \pi = \frac{243\pi}{16} m^3$$

2. In this course, a "flag" is defined as a geometric region attached to a line segment (its "pole"). An example is shown here:



a. Imagine rotating the flag about its pole and describe the resulting three-dimensional figure. Draw a picture of this figure.

The figure is a cylinder with a radius of 6 and height of 8:

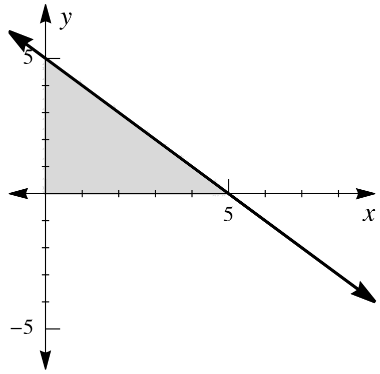


b. Calculate the volume of the rotated flag.

For a cylinder, $V = \pi r^2 h$, where r is the radius and h is the height:

$$V = \pi r^2 h = \pi \cdot 6^2 \cdot 8 = 288\pi \text{ units}^3$$

3. Examine the graph of the function $f(x) = 5 - x$:

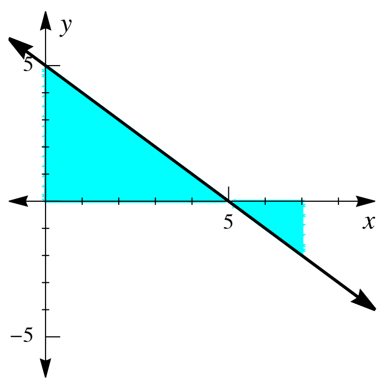


a. Calculate the area of the shaded region.

The shaded region is a triangle. The area of a triangle is $A = \frac{1}{2} b\, h$ where b is the length of the base of the triangle and h is its height. From the graph, $b = 5$ and $h = 5$:

$$A = \frac{1}{2} b\, h = \frac{1}{2} \cdot 5 \cdot 5 = \frac{25}{2} = 12.5 \text{ square units } = 12.5 \text{ un}^2$$

b. Notice that the line dips below the x-axis when $x > 5$. When you are asked to calculate the "area under a curve" this refers to the region between the curve and the x-axis. Any area *below* the x-axis is considered to be negative. Calculate the area under the curve for $0 \le x \le 7$.



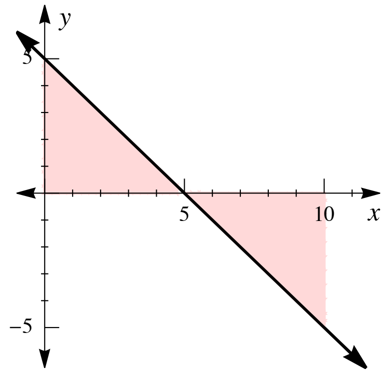
There are now two triangles: one above the x-axis with an area of 12.5 square units and one below the x-axis with a negative area of

$$A = \frac{1}{2} b\, h = \frac{1}{2} \cdot 2 \cdot 2 = 2 \text{ un}^2$$

The net area is $12.5 - 2 = 10.5 \text{ un}^2$.

c. **[Challenge]** Determine the value of k such that the area under the curve for $0 \le x \le k$ is 0.

Without graphing, symmetry can give us the answer directly: the area above the x-axis must equal the area below the x-axis. That means the bases of the two triangles must be the same ($b = 5$): $k = 2\,b = 10$. Here is the graph:

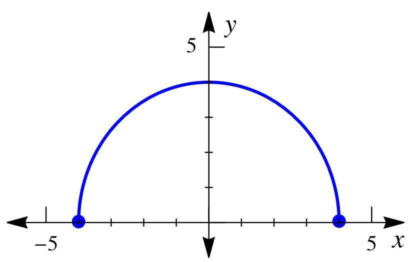


4. Sketch the function $g(x) = \sqrt{16 - x^2}$.

To sketch the function, we could use Desmos to see what it looks like, or we could notice that

$$y = \sqrt{16 - x^2} \quad \Longrightarrow \quad y^2 = 16 - x^2 \quad \Longrightarrow \quad x^2 + y^2 = 4^2$$

This last equation is the equation of circle, so $g(x)$ must be a semi-circle with radius 4 above the x-axis (because it's equation has a positive square, not a negative). Here is the graph:



a. State the domain and range of g.

The domain of the function is:

Interval Notation : $D = [-4, 4]$
Set Notation : $D = \{x : -4 \le x \le 4\}$,

The range of the function is:

Interval Notation : $R = [0, 4]$
Set Notation : $R = \{y : 0 \le y \le 4\}$

b. Use geometry to calculate the area under the curve for $0 \le x \le 4$.

The area is 1/4 of a circle and the area of a circle is $A = \pi r^2$, where r is the radius:

$$A = \frac{1}{4} \pi r^2 = \frac{1}{4} \pi \cdot 4^2 = 4 \pi \text{ un}^2$$

c. Now calculate the area under the curve for $-4 \le x \le 4$

The area is 1/2 of a circle and the area of a circle is $A = \pi r^2$, where r is the radius:

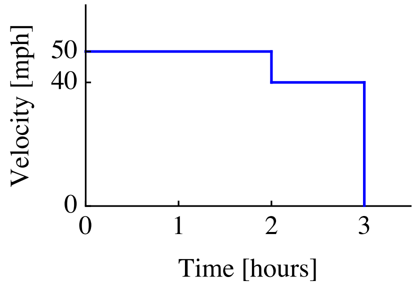
$$A = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi \cdot 4^2 = 8 \pi \text{ un}^2$$

d. What is the relationship between the answers to parts (b) and (c)?

The area in part (c) is twice the area in part (b).

5. A car travels 50 miles per hour for two hours and 40 miles per hour for one hour.

a. Sketch a graph of velocity versus time. Label the axes with units.



b. Fill out the table below for the distance versus time.

Distance is velocity (speed) times time: $s = v\,t$, which is a linear relationship: to find the distance travelled, we multiply the velocity by the time travelled. For the first two hours, we can use this relationship directly; for example, after 1.5 hours, the car has travelled

$$s = v\,t = 50\ \frac{\text{miles}}{\text{hour}} \cdot 1.5\ \text{hours} = 75\ \text{miles}$$

After two hours, the car has travelled 100 miles and the velocity (instantaneously—not sure how) becomes 40 miles per hour. So to determine the time travelled after, say 2.5 hours, we take the 100 miles travelled for the first 2 hours and add the distance travelled in the next 1/2 hour:

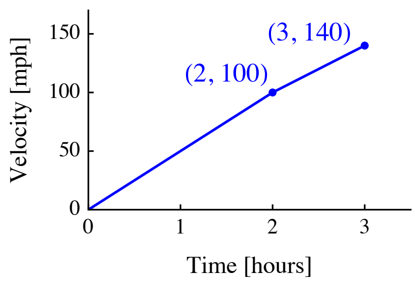
$$s = 100\ \text{miles} + \frac{1}{2}\ \text{hour} \cdot 40\ \frac{\text{miles}}{\text{hour}} = 120\ \text{miles}$$

Using these relationships, the table is:

Time [hours]	0.5	1.0	1.5	2.0	2.5	3.0
Distance [miles]	25	50	75	100	120	140

c. Sketch a graph of distance versus time. Label the axes with units.

Distance versus time is a piecewise function:



6. [Challenge] Translating Functions:

a. Graph the function $y = \frac{2}{3}x^2$. On the same set of axes graph a translation of the function that is shifted 1 unit to the right and 5 units down. Write the equation of the translated function.

To translate a function to the right h units we subtract h from x in the original equation. The function translated 1 unit to the right is:

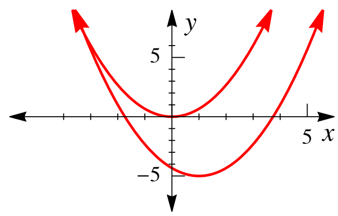
$$y = \frac{2}{3}(x-1)^2$$

To translate a function down k units, we subtract k from the original function. The function translated 5 units down is:

$$y = \frac{2}{3}x^2 - 5$$

Combining these two translations gives us:

$$y = \frac{2}{3}(x-1)^2 - 5$$



b. Does the same strategy work for $y = \frac{2}{3}x$? Write an equation that will shift $y = \frac{2}{3}x$ one unit to the right and five units down.

Yes. $y = \frac{2}{3}(x - 1) - 5$

c. Compare the graphs of $y = -\frac{1}{2}x$ and $y = -\frac{1}{2}(x + 2) + 3$. Describe their similarities and differences.

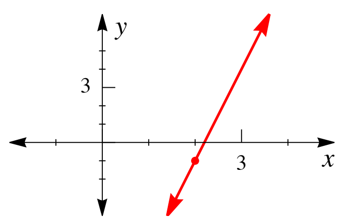
They are both equations of lines with a slope of $-1/2$. The second line is shifted 2 units to the left and up 3 units.

d. Explain how you know that the graph of $y = -9(x + 1) - 6$ goes through the point $(-1, -6)$ and has a slope of -9.

The graph of $y = -9x$ goes through the origin, $(0, 0)$. The graph of the given function is translated to the left 1 unit and down 6 units, so it must go through $(-1, -6)$.

e. Sketch the graph of $y = 5(x - 2) - 1$.

We know the graph is a line with slope of 5 and must go through the point $(2, -1)$.



7. **[Challenge]** Write the equation of the line through the point (–5, –2) with a slope of –3 in graphing form using the method developed in Problem 6.

If we start with $y = -3x$ and translate to the left 5 units and down 2 units the equation becomes: $y = -3(x + 5) - 2$.

8. **[Challenge]** Now you know *two* general equations used to write the equation of a line:

$y = mx + b$ and $y = m(x - h) + k$

Under what circumstances is each equation easier to use? For parts (a) through (c) below, determine which method is best to use with the given information. Then, write the equation of the line.

The first equation (*slope-intercept* form) is easiest when we know the slope and the y-intercept. The second equation (*point-slope* form) is easiest when we know one point the line goes through and its slope.

a. $m = -\frac{2}{5}$ and passes through $(-6, 2)$

Point-slope form: $y = -\frac{2}{5}(x - (-6)) + 2 = -\frac{2}{5}(x + 6) + 2$

b. $m = 3$ and $b = -6$

Slope-intercept form: $y = 3x + (-6) = 3x - 6$

c. passes through $(2, 8)$ and $(1, 3)$

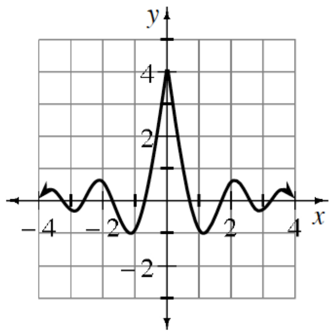
Point-slope form:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 8}{1 - 2} = \frac{-5}{-1} = 5$$

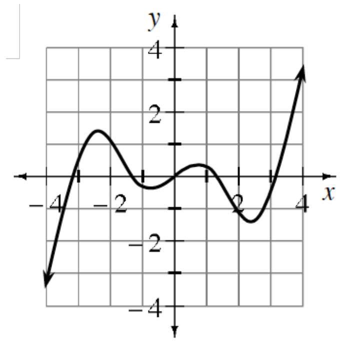
$y = 5(x - 2) + 8$ or $y = 5(x - 1) + 3$

9. **[Challenge]** For each function sketched below, sketch $y = f(-x)$ and compare it with the original graph. Then describe its symmetry.

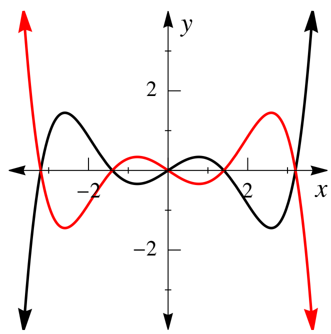
a.



b.



For (a) the graph of $f(-x)$ is identical to the original graph and it is symmetric with respect to the y-axis (reflection across the y-axis). For (b) the graph is reflected across the x-axis and it is symmetric with respect to the origin (reflection through the origin, which is also 180° rotational symmetry about the origin).



Even and Odd Functions—Informally: A function that is symmetric with respect to the y-axis, like the one in part (a) above, is called an *even function*. A function that is symmetric with respect to the origin, like the one in part (b), is called an *odd function*. Sketch more examples of even and odd functions. Include how you can test whether a function is even or odd. Then list some famous even/odd functions that you have studied in a previous course, and the symmetries associated with even and odd functions.

Even: symmetric about the y-axis, the height at an x-value is the same height at the $-x$-value, $y = x^2$ is an example.

Odd: symmetric about the origin, the height at an x-value is the opposite height at the $-x$-value, $y = x^3$ is an example.