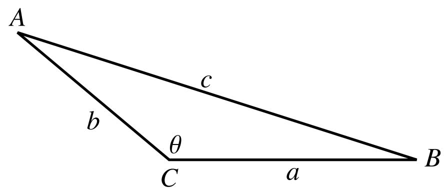
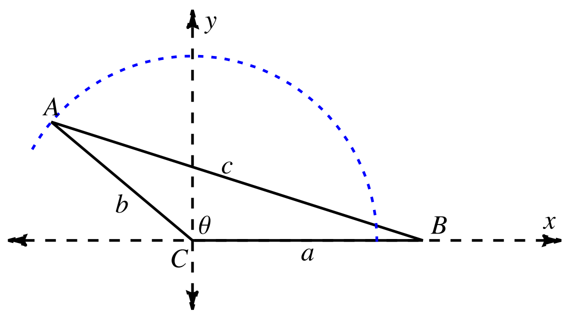
Trigonometry 15: The Law of Cosines

Let's take an oblique triangle that is labeled like this:



and put it on a Cartesian Coordinate plane (x- and y-axis):



From this graph we know

- Vertex C is at the origin so its coordinate points are $(0, 0)$.
- Vertex B is at $(a, 0)$.
- Vertex A is at some point, (x, y)

If we think of vertex A as being on a circle with radius b (shown as the blue dashed arc), then by the definitions of sine and cosine, the coordinates of vertex A must

- $x = b\cos\theta$
- $y = b\sin\theta$

Since length c is the distance from A to B, we can use the Distance Formula to derive an expression for c:

$$\begin{aligned} c^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (x - a)^2 + (y - 0)^2 \\ &= (b\cos\theta - a)^2 + (b\sin\theta)^2 \\ &= b^2\cos^2\theta - 2ab\cos\theta + a^2 + b^2\sin^2\theta \\ &= a^2 + b^2\left(\cos^2\theta + \sin^2\theta\right) - 2ab\cos\theta \qquad \left(\cos^2\theta + \sin^2\theta = 1\right) \\ c^2 &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

This is known as the **law of cosines**. Notice that when $\theta = 90°$, the Law of Cosines becomes the Pythagorean Theorem!

This law can be very helpful for finding a missing side length in any triangle (as long as θ is the angle between the known sides). For example, find the missing side length in the following triangle: