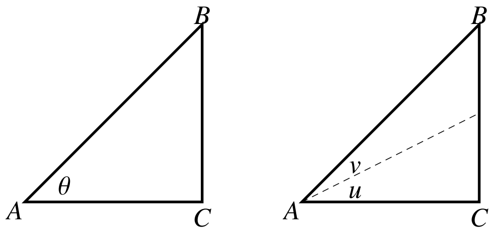
Trigonometry 36: Sum Formulas, Part I

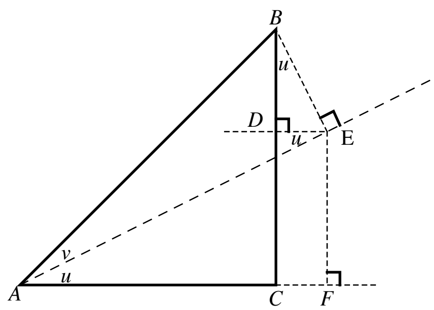
Let's take a simple right triangle, ABC, with an angle θ that is equal to the sum of u and v:



Using our trig formulas for right triangles we know:

$$\sin(\theta) = \sin(u + v) = \frac{O}{H} = \frac{\text{BC}}{\text{AB}}$$

For reasons that will become clearer later, let's extend the dashed line and find other angles that have a measure equal to u:



$$\sin(u + v) = \frac{O}{H} = \frac{\text{BC}}{\text{AB}} = \frac{\text{CD} + \text{BD}}{\text{AB}} = \frac{\text{EF} + \text{BD}}{\text{AB}} = \frac{\text{EF}}{\text{AB}} + \frac{\text{BD}}{\text{AB}}$$

Looking carefully, we can use the various triangles in the diagram to obtain some equalities:

Triangle	Equality
AEF	$\sin(u) = \frac{\text{EF}}{\text{AE}}$
AEB	$\sin(v) = \frac{\text{BE}}{\text{AB}}$
BDE	$\cos(u) = \frac{\text{BD}}{\text{BE}}$
AEB	$\cos(v) = \frac{\text{AE}}{\text{AB}}$

We a little manipulation, we can obtain one of the most important formulas in trigonometry, the **sum formula** for sine:

$$\sin(u + v) = \frac{\text{EF}}{\text{AB}} + \frac{\text{BD}}{\text{AB}} = \frac{\text{EF}}{\text{AB}} \cdot \frac{\text{AE}}{\text{AE}} + \frac{\text{BD}}{\text{AB}} \cdot \frac{\text{BE}}{\text{BE}} = \frac{\text{EF}}{\text{AE}} \cdot \frac{\text{AE}}{\text{AB}} + \frac{\text{BD}}{\text{BE}} \cdot \frac{\text{BE}}{\text{AB}}$$

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

One of the uses for the sum formula is two find exact values for sine that are not on the unit circle, for example,

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\,\frac{\sqrt{2}}{2} + \frac{1}{2}\,\frac{\sqrt{2}}{2} = \frac{\sqrt{2}\left(1 + \sqrt{3}\right)}{2\sqrt{2}\,\sqrt{2}} = \frac{1 + \sqrt{3}}{2\sqrt{2}}$$