Trigonometry 39: Double-Angle and Half-Angle Formulas

The sum formula for sine can also be used to derive the **double-angle** formula for sine:

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
$$\sin(\theta+\theta) = \sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

The sum formula for cosine can also be used to derive the double-angle formula for cosine:

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$
$$\cos(\theta+\theta) = \cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

This double-angle formula can then be used to derive the **power-reducing** and **half-angle formulas** for cosine:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = \cos^2\theta - \left(1-\cos^2\theta\right)$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$
$$\cos\theta = \sqrt{\frac{\cos 2\theta + 1}{2}} \qquad \text{or} \qquad \cos\beta/2 = \sqrt{\frac{\cos\beta + 1}{2}}$$

Half-angle formulas are very useful for finding exact values of trig functions that are not on our original unit circle:

$$\cos(\pi/12) = \sqrt{\frac{\cos\pi/6 + 1}{2}} = \frac{1}{\sqrt{2}}\sqrt{\frac{\sqrt{3}}{2} + \frac{2}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

Nested radicals can be very difficult to **de-nest**, but it is possible to show this is the same result we got on the previous handout.

We can continue finding cosine values indefinitely,

$$\cos(\pi/24) = \sqrt{\frac{\cos\pi/12 + 1}{2}} = \frac{1}{\sqrt{2}}\sqrt{\frac{\sqrt{2+\sqrt{3}}}{2} + \frac{2}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}$$

The double-angle formula for cosine can also be used to derive the **power-reducing** and **half-angle formulas** for sine:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = \left(1-\sin^2\theta\right) - \sin^2\theta$$
$$\cos 2\theta == 1 - 2\sin^2\theta$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \qquad \text{or} \qquad \sin\beta/2 = \sqrt{\frac{1-\cos\beta}{2}}$$

There are sum, difference, power-reducing, and half-angle formulas for the tangent function, but are outside our scope.