

Trigonometry 40: Product-to-Sum Formulas

The last set of formulas we will look at are the **product-to-sum formulas**, which are also useful for solving trig problems. Let's start with the sum formula for cosine:

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

We can rearrange this to get

$$\sin u \sin v = \cos u \cos v - \cos(u + v)$$

Now let's look at the difference formula for cosine and rearrange it:

$$\cos(u - v) = \cos u \cos v + \sin u \sin v \quad \implies \quad \sin u \sin v = \cos(u - v) - \cos u \cos v$$

When we add these two expressions for $\sin u \sin v$ together we get

$$2 \sin u \sin v = \cos u \cos v - \cos(u + v) + \cos(u - v) - \cos u \cos v$$

$$2 \sin u \sin v = \cos(u - v) - \cos(u + v)$$

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

Notice how this formula converts a multiplication (product) into addition/subtraction.

A similar technique can be used to derive the other three product to sum formulas:

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$